

# Instrumental variables: Experiments with non-compliance

INFO/STSCI/ILRST 3900: Causal Inference

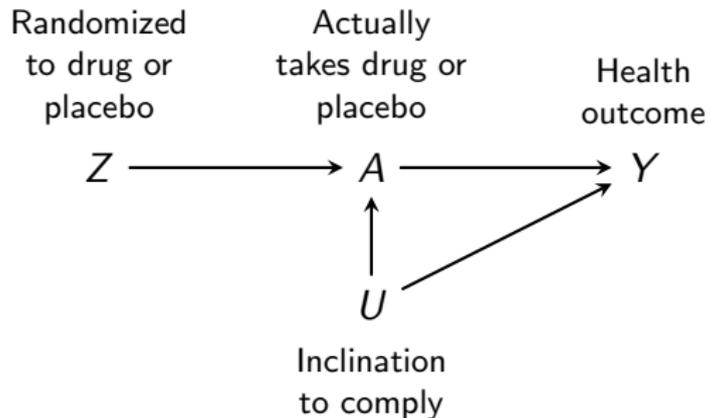
17 Oct 2023

# Learning goals for today

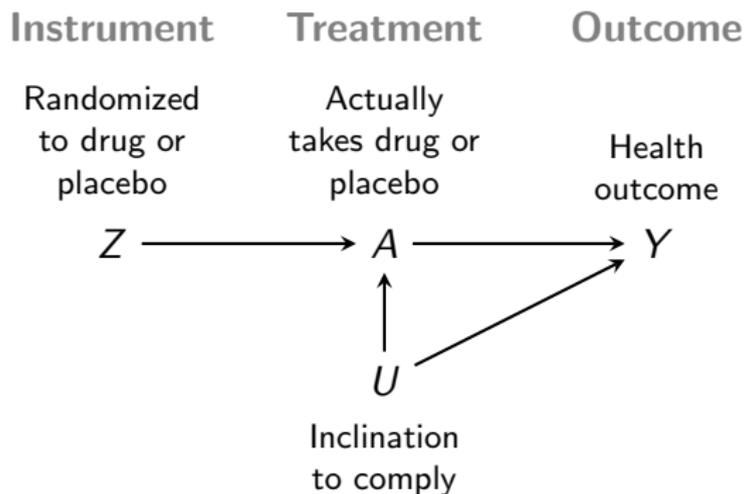
At the end of class, you will be able to:

1. Understand the logic of instrumental variables
2. Derive the average effect among compliers in experiments with noncompliance

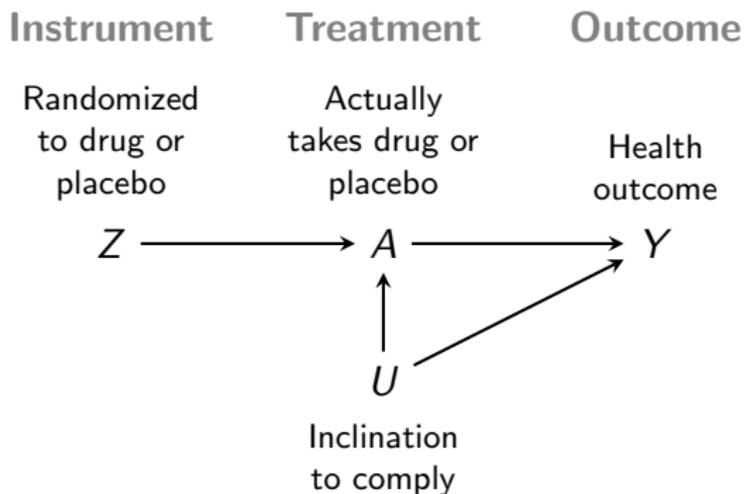
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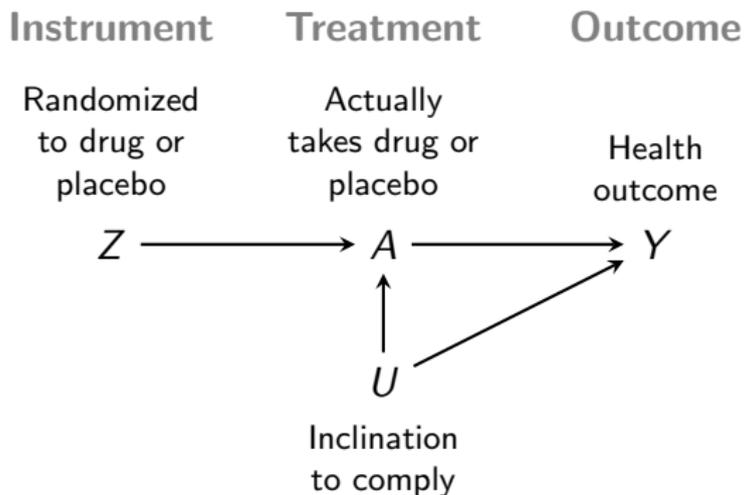
# Instrumental variables: Experiment with noncompliance



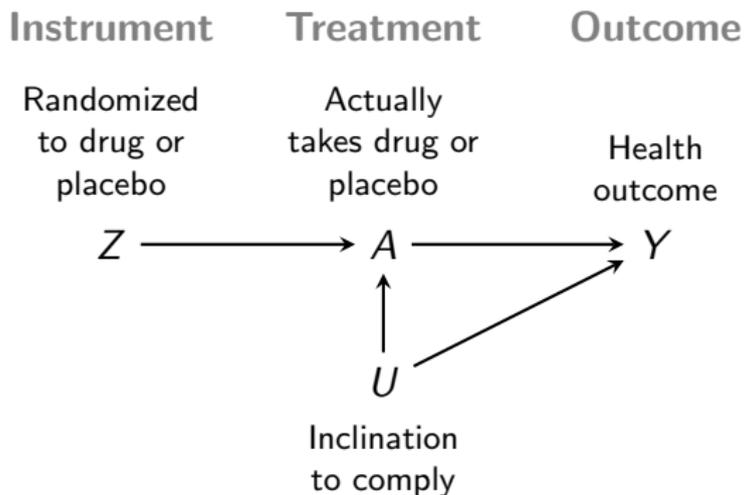
Two ideas

1. Intent to treat effect
2. Average effect among compliers

# Instrumental variables: 1) Intent to treat effect

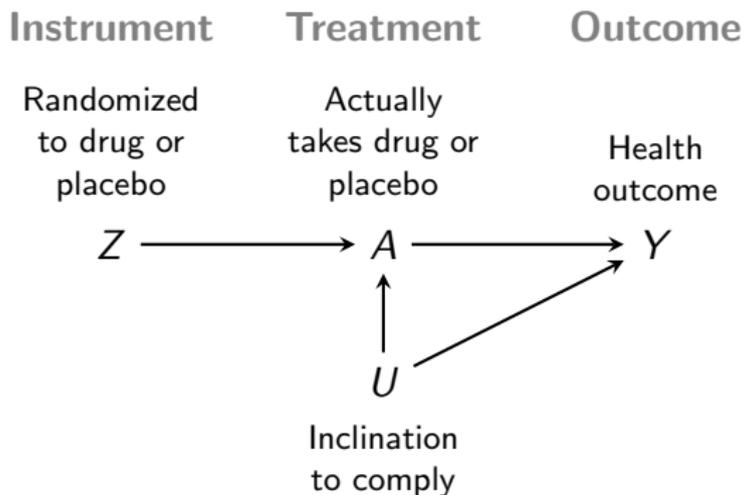


# Instrumental variables: 1) Intent to treat effect



Ignore  $A$ . What is the effect of  $Z$  on  $Y$ ?

# Instrumental variables: 1) Intent to treat effect

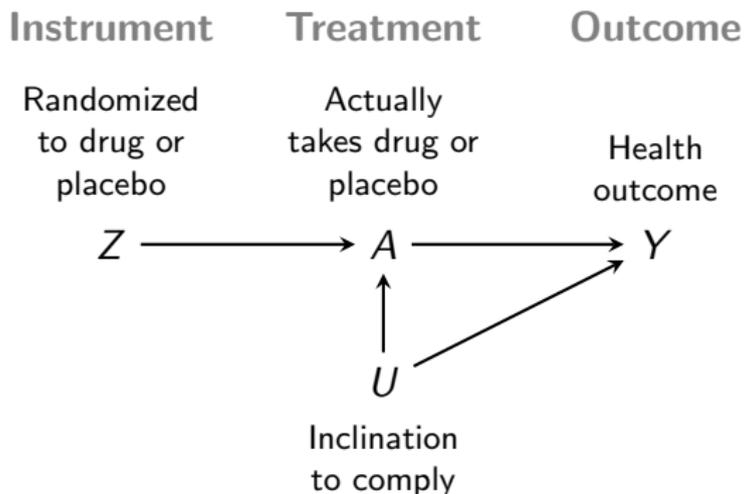


Ignore A. What is the effect of Z on Y?

$$E(Y^1 - Y^0)$$

Outcome under  $Z = 1$       Outcome under  $Z = 0$

# Instrumental variables: 1) Intent to treat effect



Ignore  $A$ . What is the effect of  $Z$  on  $Y$ ?

$$E(Y^1 - Y^0) =$$

↑            ↑

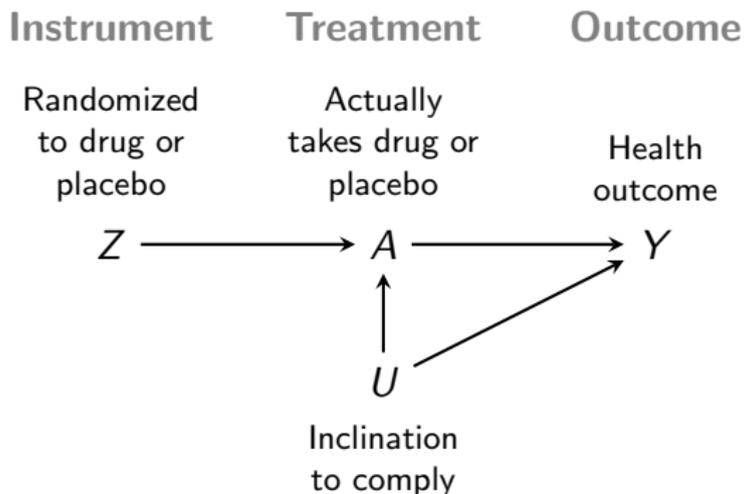
Outcome    Outcome

under        under

$Z = 1$        $Z = 0$

By  
Positivity  
Consistency  
Exchangeability  
for  $Z$

# Instrumental variables: 1) Intent to treat effect



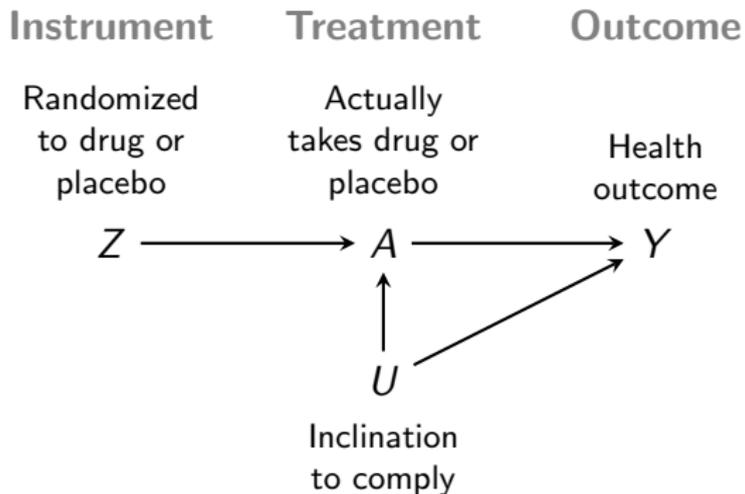
Ignore  $A$ . What is the effect of  $Z$  on  $Y$ ?

$$\begin{array}{ccc} E(Y^1 - Y^0) & = & E(Y | Z = 1) - E(Y | Z = 0) \\ \begin{array}{c} \uparrow \\ \text{Outcome} \\ \text{under} \\ Z = 1 \end{array} & & \begin{array}{c} \uparrow \\ \text{By} \\ \text{Positivity} \\ \text{Consistency} \\ \text{Exchangeability} \\ \text{for } Z \end{array} & & \begin{array}{c} \text{Mean difference in} \\ \text{observable outcomes} \end{array} \\ \begin{array}{c} \uparrow \\ \text{Outcome} \\ \text{under} \\ Z = 0 \end{array} & & & & \end{array}$$

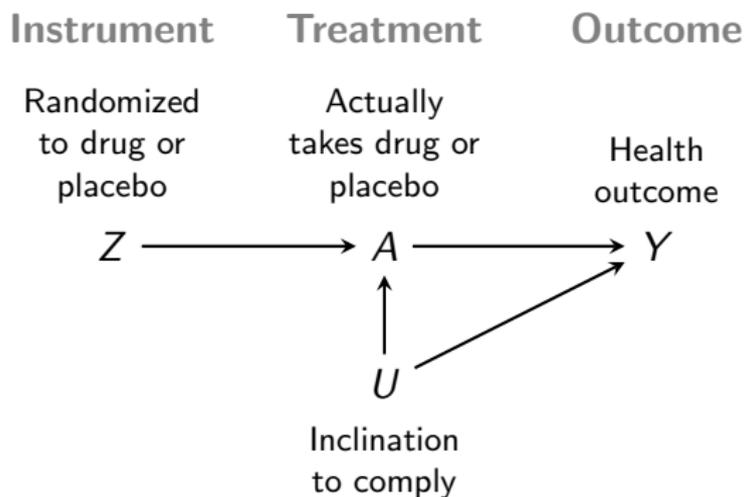
## Instrumental variables: 2) Average effect among compliers

Imbens, G., & Angrist, J. (1994). [Identification and estimation of local average treatment effects](#). *Econometrica*, 62(2), 467-475.

## Instrumental variables: 2) Average effect among compliers

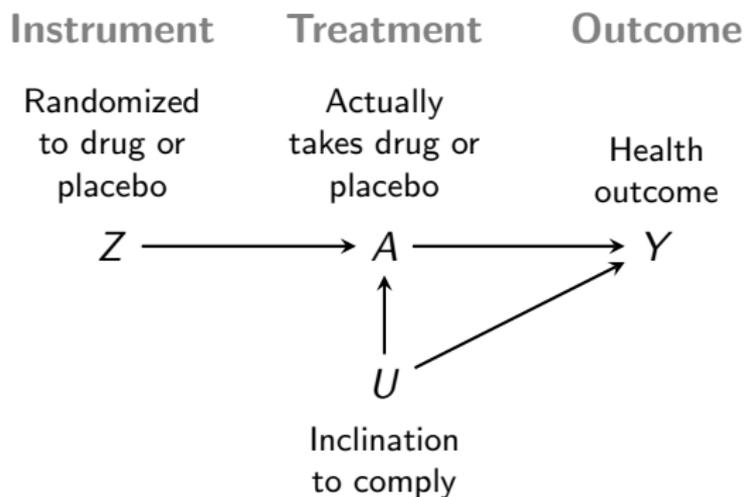


## Instrumental variables: 2) Average effect among compliers



**Key insight:** The effect of  $Z$  on  $Y$  operates entirely through  $A$

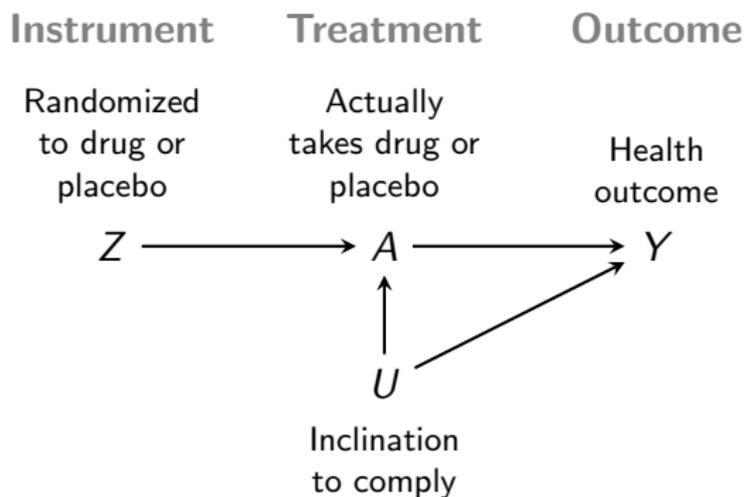
## Instrumental variables: 2) Average effect among compliers



**Key insight:** The effect of  $Z$  on  $Y$  operates entirely through  $A$

1. Study the effect of  $Z \rightarrow Y$

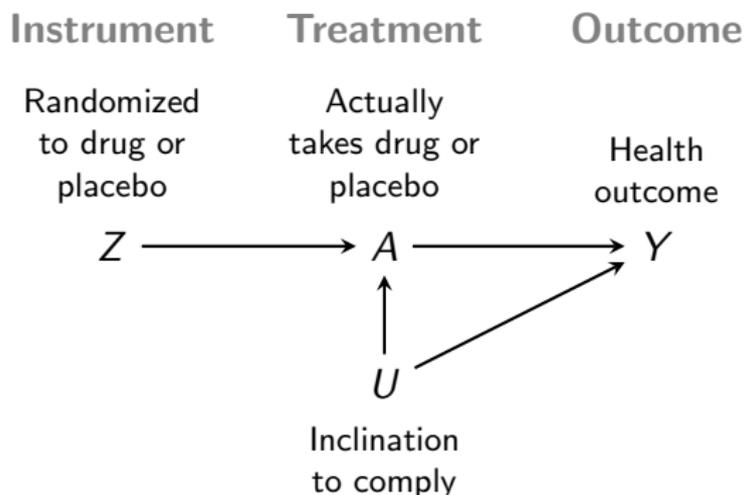
## Instrumental variables: 2) Average effect among compliers



**Key insight:** The effect of  $Z$  on  $Y$  operates entirely through  $A$

1. Study the effect of  $Z \rightarrow Y$  (we just did)

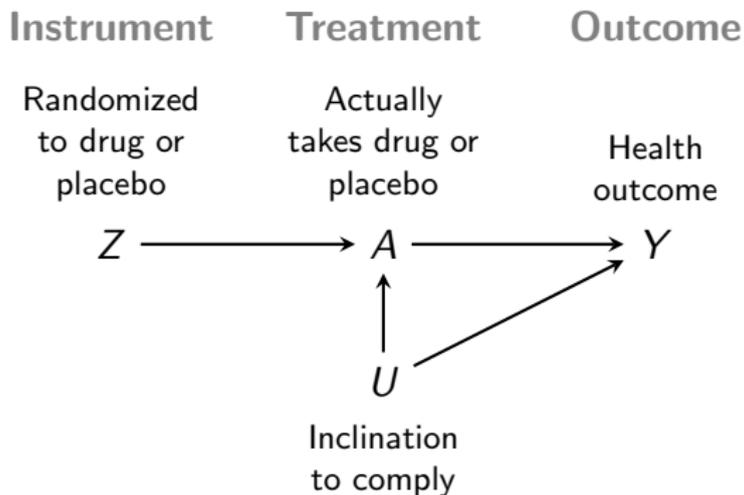
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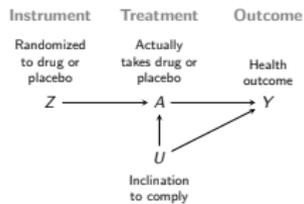
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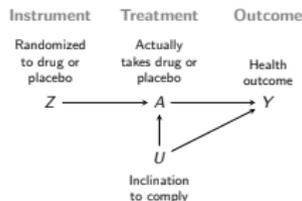
**Key insight:** The effect of  $Z$  on  $Y$  operates entirely through  $A$

1. Study the effect of  $Z \rightarrow Y$  (we just did)
2. Study the effect of  $Z \rightarrow A$
3. Learn about  $A \rightarrow Y$  since  $Z \rightarrow Y$  is  $Z \rightarrow A \rightarrow Y$

## Instrumental variables: 2) Average effect among compliers

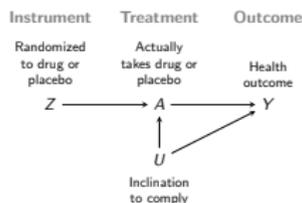


## Instrumental variables: 2) Average effect among compliers



The effect  $Z \rightarrow A$  has four **principal strata**:  
latent sets of people who respond to  $Z$  a particular way

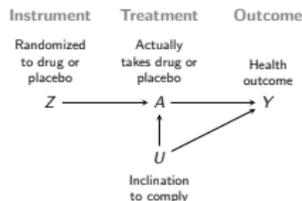
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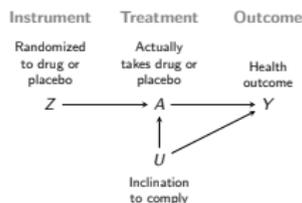


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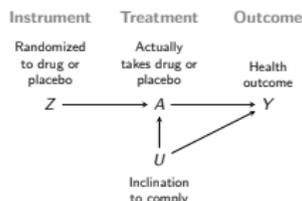
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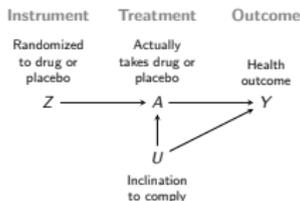
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Defiers	$A^{Z=0} = 1$	$A^{Z=1} = 0$	(defy assignment)

## Instrumental variables: 2) Average effect among compliers

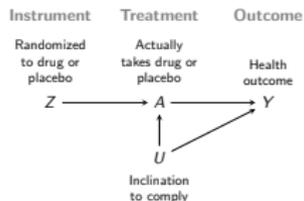


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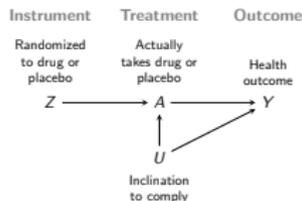
**Discuss:** In which strata is the effect  $Z \rightarrow Y$  zero?

## Instrumental variables: 2) Average effect among compliers



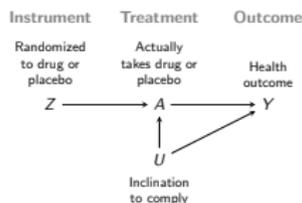
Among **always takers** and **never takers**,

## Instrumental variables: 2) Average effect among compliers



Among **always takers** and **never takers**,  
 $Z$  does not affect  $A$

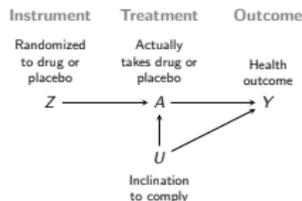
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Among **always takers** and **never takers**,  
 $Z$  does not affect  $A$

$Z$  only affects  $Y$  through  $A$

## Instrumental variables: 2) Average effect among compliers

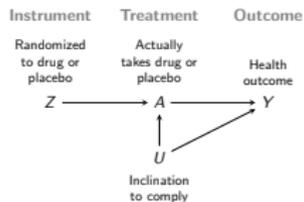


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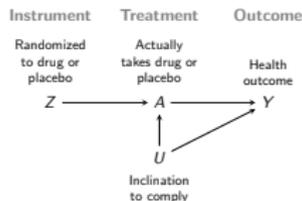
In these strata,  $Z$  does not affect  $Y$

## Instrumental variables: 2) Average effect among compliers



Among **compliers**,

## Instrumental variables: 2) Average effect among compliers

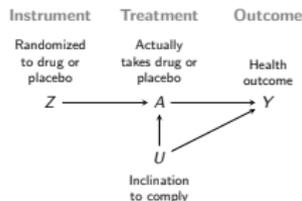


Among **compliers**,

$Z = 1$  implies  $A = 1$  and

$Z = 0$  implies  $A = 0$

## Instrumental variables: 2) Average effect among compliers



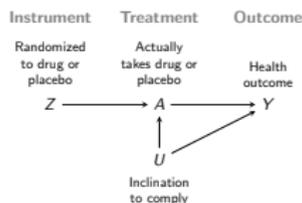
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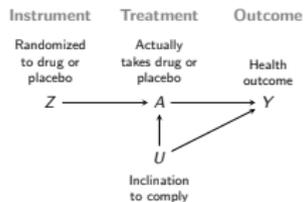
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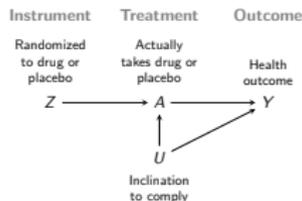
$Z \rightarrow Y$  and  $A \rightarrow Y$  are the same

## Instrumental variables: 2) Average effect among compliers



Among **defiers**,

## Instrumental variables: 2) Average effect among compliers

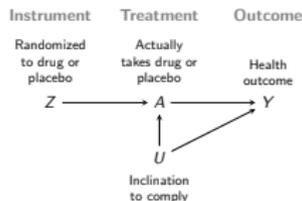


Among **defiers**,

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$Z = 0$  implies  $A = 1$

## Instrumental variables: 2) Average effect among compliers



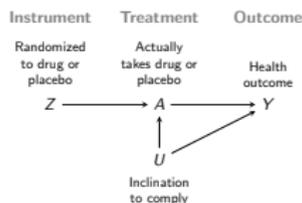
Among **defiers**,

$Z = 1$  implies  $A = 0$  and

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In these strata,  $Z = 1 - A$

## Instrumental variables: 2) Average effect among compliers



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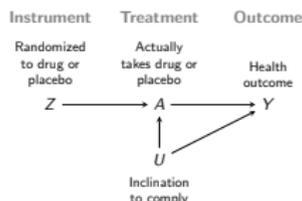
$Z = 1$  implies  $A = 0$  and

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In these strata,  $Z = 1 - A$

$Z \rightarrow Y$  and  $A \rightarrow Y$  are the same magnitude  
but have opposite signs

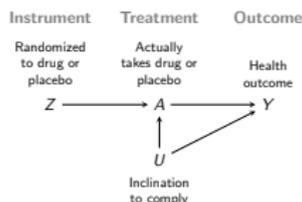
## Instrumental variables: 2) Average effect among compliers



### Four principal strata

Compliers	$(Z \rightarrow A) = +1$	$(Z \rightarrow Y) = (A \rightarrow Y)$
Always takers	$(Z \rightarrow A) = 0$	$(Z \rightarrow Y) = 0$
Never takers	$(Z \rightarrow A) = 0$	$(Z \rightarrow Y) = 0$
Defiers	$(Z \rightarrow A) = -1$	$(Z \rightarrow Y) = -(A \rightarrow Y)$

## Instrumental variables: 2) Average effect among compliers

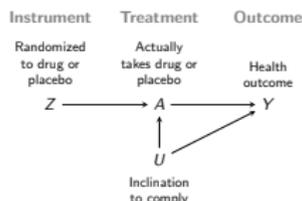


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Assume **no defiers** in the population

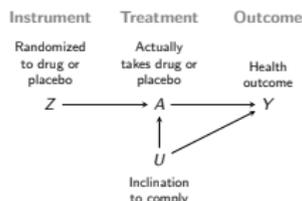
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## Instrumental variables: 2) Average effect among compliers

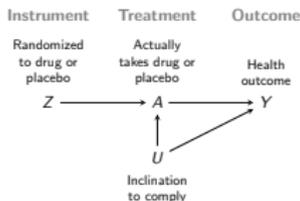


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### Discuss a hypothetical.

## Instrumental variables: 2) Average effect among compliers



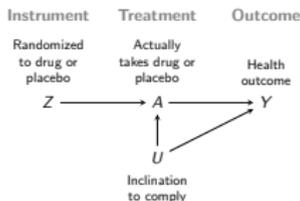
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### Discuss a hypothetical.

Population is 50% compliers, 25% always takers, 25% never takers

## Instrumental variables: 2) Average effect among compliers



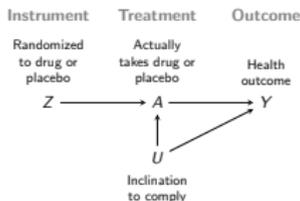
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### Discuss a hypothetical.

Population is 50% compliers, 25% always takers, 25% never takers  
Average effect of  $Z \rightarrow Y$  among compliers is 0.6

## Instrumental variables: 2) Average effect among compliers



### Four principal strata

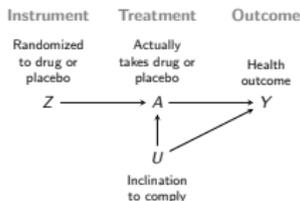
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### Discuss a hypothetical.

Population is 50% compliers, 25% always takers, 25% never takers  
Average effect of  $Z \rightarrow Y$  among compliers is 0.6

What is the average effect of  $Z \rightarrow Y$  in the population?

## Instrumental variables: 2) Average effect among compliers



### Four principal strata

Compliers	$(Z \rightarrow A) = +1$	$(Z \rightarrow Y) = (A \rightarrow Y)$
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### Discuss a hypothetical.

Population is 50% compliers, 25% always takers, 25% never takers  
Average effect of  $Z \rightarrow Y$  among compliers is 0.6

What is the average effect of  $Z \rightarrow Y$  in the population? **0.3**

Instrumental variables: 2) Average effect among compliers

Deriving the general case:

## Instrumental variables: 2) Average effect among compliers

Deriving the general case:

$$E(Y^{Z=1} - Y^{Z=0})$$

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Deriving the general case:

$$\begin{aligned} & E(Y^{Z=1} - Y^{Z=0}) \\ &= \sum_s E(Y^{Z=1} - Y^{Z=0} \mid S = s) \underbrace{P(S = s)}_{\substack{\text{Denote} \\ \pi_s}} \end{aligned}$$

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Deriving the general case:

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## Instrumental variables: 2) Average effect among compliers

Deriving the general case:

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Instrumental variables: 2) Average effect among compliers

$$E(Y^{Z=1} - Y^{Z=0}) = E(Y^{Z=1} - Y^{Z=0} \mid S = \text{Complier})\pi_{\text{Complier}}$$

## Instrumental variables: 2) Average effect among compliers

Average effect of instrument = Average among compliers  $\times$  Proportion who comply

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Among compliers,  $(Z \rightarrow Y) = (A \rightarrow Y)$ .

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## Instrumental variables: 2) Average effect among compliers

$$\begin{array}{l} \text{Average effect} \\ \text{of instrument} \end{array} = \begin{array}{l} \text{Average among} \\ \text{compliers} \end{array} \times \begin{array}{l} \text{Proportion} \\ \text{who comply} \end{array}$$

$$E(Y^{Z=1} - Y^{Z=0}) = E(Y^{Z=1} - Y^{Z=0} \mid S = \text{Complier})\pi_{\text{Complier}}$$

Among compliers,  $(Z \rightarrow Y) = (A \rightarrow Y)$ .

$$E(Y^{Z=1} - Y^{Z=0}) = E(Y^{A=1} - Y^{A=0} \mid S = \text{Complier})\pi_{\text{Complier}}$$

Rearrange to get the complier average treatment effect

## Instrumental variables: 2) Average effect among compliers

Average effect of instrument = Average among compliers  $\times$  Proportion who comply

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$$\begin{aligned} E(Y^{A=1} - Y^{A=0} \mid S = \text{Complier}) &= \frac{E(Y^{Z=1} - Y^{Z=0})}{\pi_{\text{Complier}}} \\ &= \frac{E(Y^{Z=1} - Y^{Z=0})}{E(A^{Z=1} - A^{Z=0})} \end{aligned}$$

# Learning goals for today

At the end of class, you will be able to:

1. Understand the logic of instrumental variables
2. Derive the average effect among compliers in experiments with noncompliance