Parametric g-Formula

Cornell STSCI / INFO / ILRST 3900
Fall 2023
causal3900.github.io

28 Sep 2023

Learning goals for today

At the end of class, you will be able to

- estimate average causal effects with a parametric model
  - for the outcome $\text{E}(Y \mid A, \vec{L})$
  - for the treatment $\text{P}(A \mid \vec{L})$

After class:

- Hernán and Robins 2020 Chapter 12.1–12.5, 13, 15.1

Nonparametric estimation

Causal assumptions

$$\begin{tikzcd}
\vec{L} \arrow[r] \arrow[rr, bend left] & A \arrow[r] & Y
\end{tikzcd}$$

Nonparametric estimator

$$\hat{\mathrm{E}}(Y^a) = \frac{1}{n} \sum_{i=1}^{n} \hat{\mathrm{E}}(Y \mid \vec{L} = \vec{\ell}_i, A = a)$$

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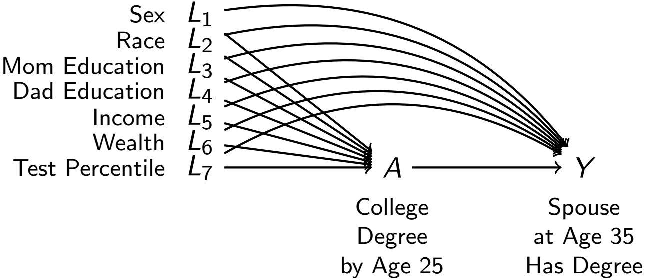
For every unit i,

- find units who look like them on confounders $\vec{L}$
- who actually got treatment $A = a$
- take the average among those units

Then average over all units

Nonparametric estimation breaks down



Sex L_1

Race L_2

Mom Education L_3

Dad Education L_4

Income L_5

Wealth L_6

Test Percentile L_7

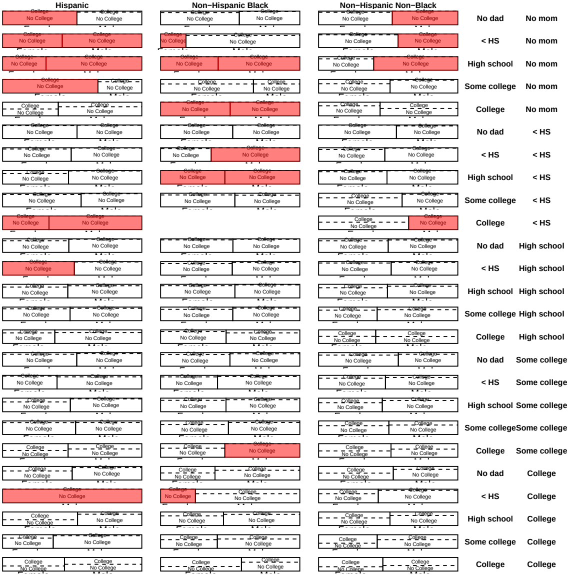
A

Y

College
Degree
by Age 25

Spouse
at Age 35
Has Degree

Nonparametric estimation breaks down



Parametric estimation: Outcome model

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First, learn a model to predict Y given $\vec{L}$ and A

$$\hat{\text{E}}(Y \mid \vec{L}, A) = \hat{\alpha} + \vec{L}'\hat{\vec{\gamma}} + A\hat{\beta}$$

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For every unit i,

- change the treatment value to a
- predict the outcome

Then average over all units

The parametric g-formula: Connection to $\hat{\beta}$

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Estimator for the effect $\mathsf{E}(Y^1) - \mathsf{E}(Y^0)$:

The parametric g-formula: Connection to $\hat{\beta}$

Estimator for the effect $\mathrm{E}(Y^1) - \mathrm{E}(Y^0)$:

$$\hat{\mathrm{E}}(Y^1) - \hat{\mathrm{E}}(Y^0) = \left(\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\alpha} + \hat{\gamma}\ell_i + \hat{\beta} \times 1\right)\right) - \left(\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\alpha} + \hat{\gamma}\ell_i + \hat{\beta} \times 0\right)\right)$$

The parametric g-formula: Connection to $\hat{\beta}$

Estimator for the effect $\mathrm{E}(Y^1) - \mathrm{E}(Y^0)$:

$$
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&= \hat{\beta}
\end{aligned}
$$

With OLS, the parametric g-formula collapses on the coefficient.

The parametric g-formula is more general

Suppose the effect of A depends on L

$$\mathsf{E}(Y \mid A, L) = \alpha + \gamma L + \beta A + \eta A L$$

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&= \frac{1}{n}\sum_{i=1}^{n}\left(\hat\beta + \hat\eta \ell_i\right)
\end{aligned}$$

The g-formula no longer collapses to a coefficient!

Parametric g-formula: Outcome model recap

$$\begin{tikzcd}
\vec{L} \arrow[r] \arrow[rr, bend left] & A \arrow[r] & Y
\end{tikzcd}$$

1. Model the outcome mean $\mathrm{E}(Y \mid A, \vec{L})$
2. Change everyone's treatment to the value of interest
3. Predict for everyone
4. Take the average

$$\hat{\mathrm{E}}(Y^a) = \frac{1}{n} \sum_{i=1}^{n} \hat{\mathrm{E}}(Y \mid \vec{L} = \vec{\ell}_i, A = a)$$

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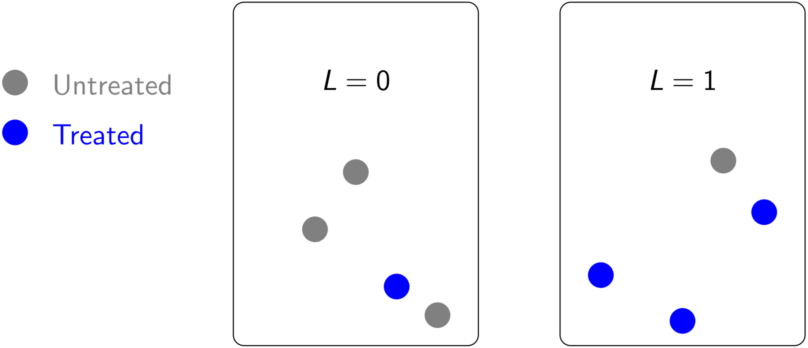
Inverse probability of treatment weighting



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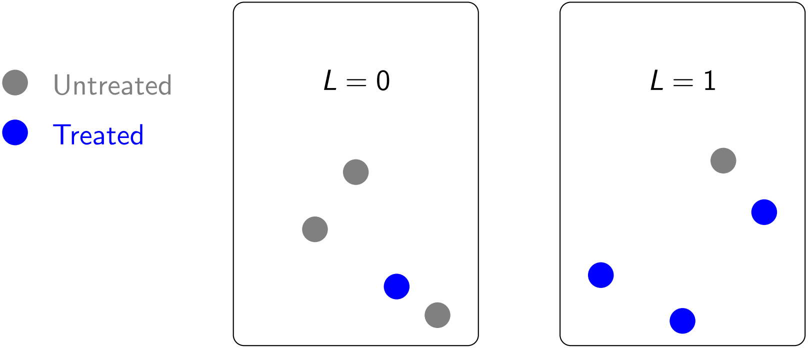
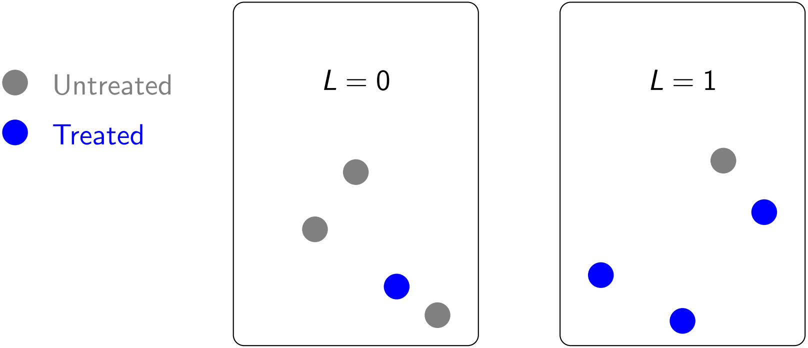


Propensity score: $\pi_i = \mathrm{P}(A = A_i \mid L = L_i)$

Inverse probability of treatment weighting



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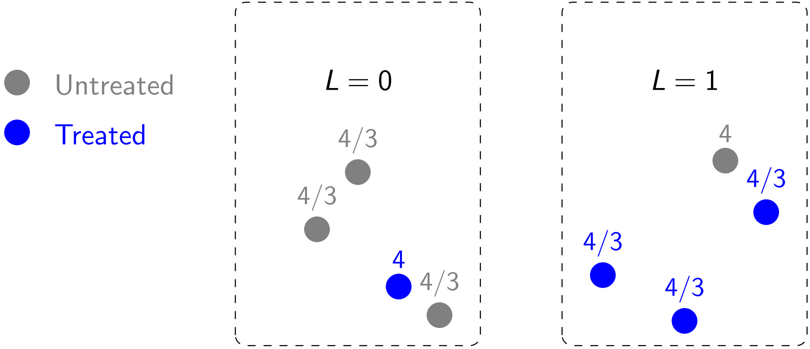

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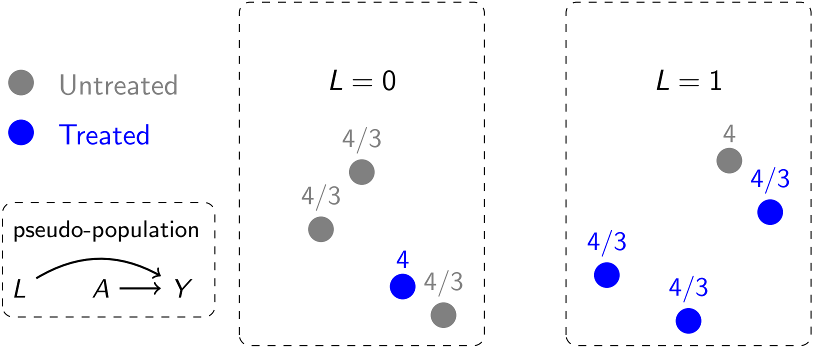

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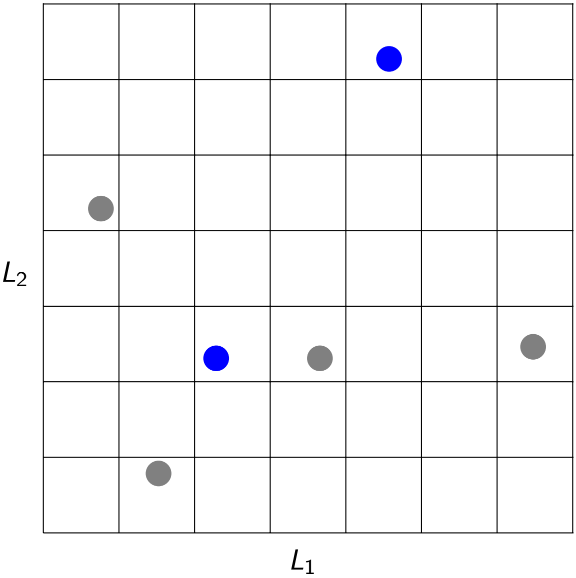
L_2
L_1

Model the treatment assignment

$$\hat{\mathrm{P}}(A = 1 \mid \vec{L}) = \text{logit}^{-1}\left(\hat{\alpha} + \hat{\vec{\gamma}}\vec{L}\right)$$

Predict the propensity score for each unit

$$\hat{\pi}_i = \begin{cases} \text{logit}^{-1}\left(\hat{\alpha} + \hat{\vec{\gamma}}\vec{L}\right) & \text{if } A_i = 1 \\ 1 - \text{logit}^{-1}\left(\hat{\alpha} + \hat{\vec{\gamma}}\vec{L}\right) & \text{if } A_i = 0 \end{cases}$$

Estimate by inverse probability weighting

$$\hat{\mathrm{E}}(Y^a) = \frac{1}{N}\sum_{i:A_i=a}\frac{Y_i}{\hat{\pi}_i}$$

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Two solutions

1. Trim the weights
2. Truncate the weights

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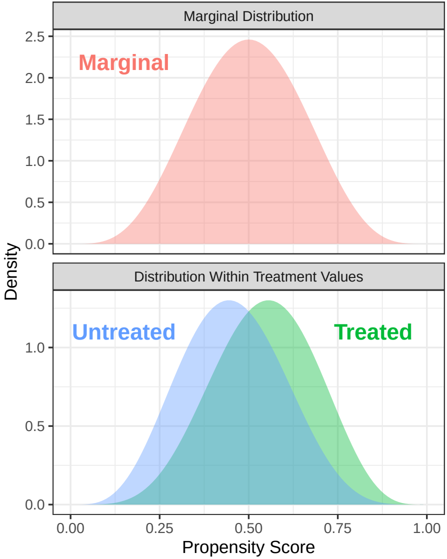
Both solutions accept bias in order to reduce variance

Accepting bias to reduce variance: Trimming



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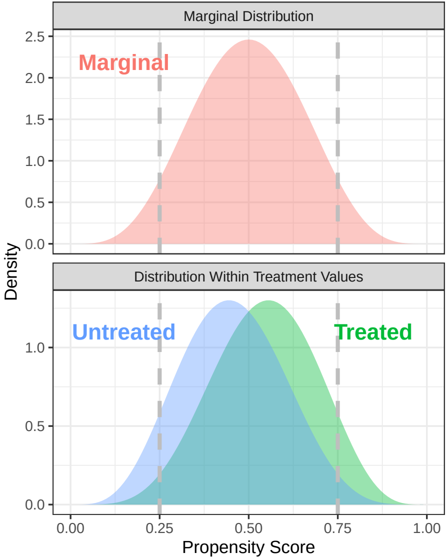

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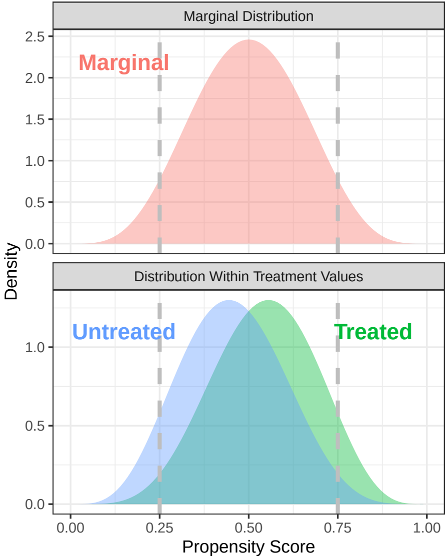
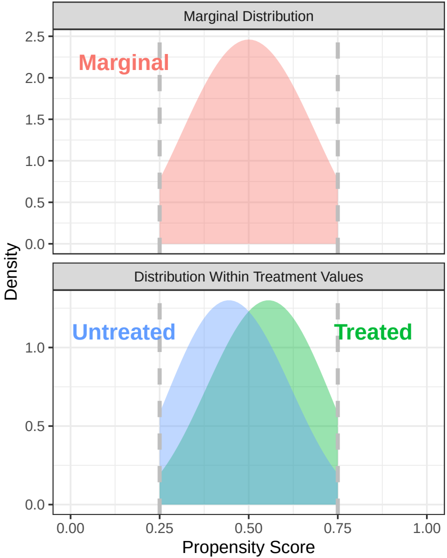


Drop units with extreme weights

Accepting bias to reduce variance: Trimming



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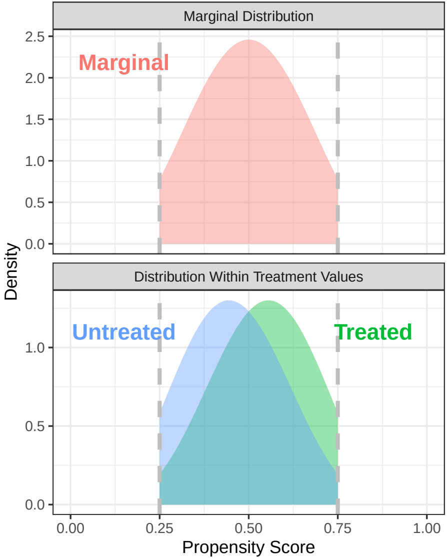
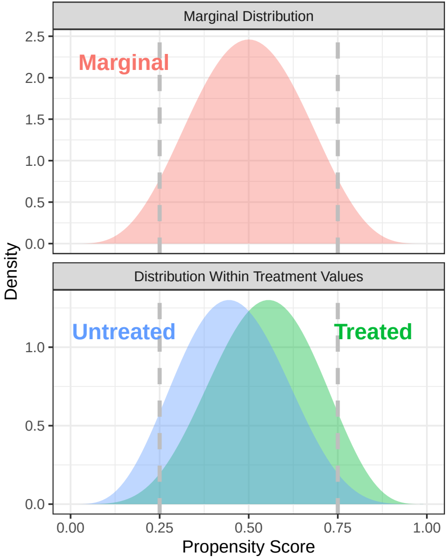


Drop units with
extreme weights

Changes target population
— Biased for full population

Accepting bias to reduce variance: Weight truncation



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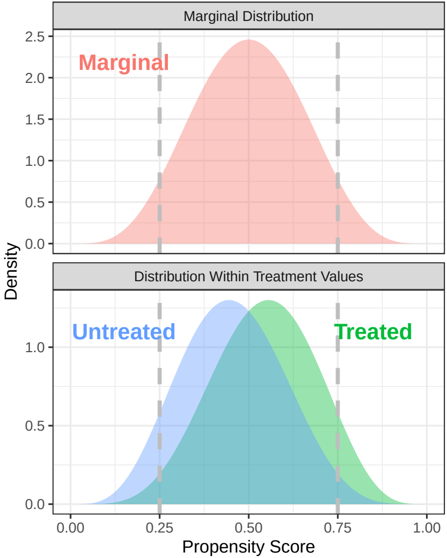
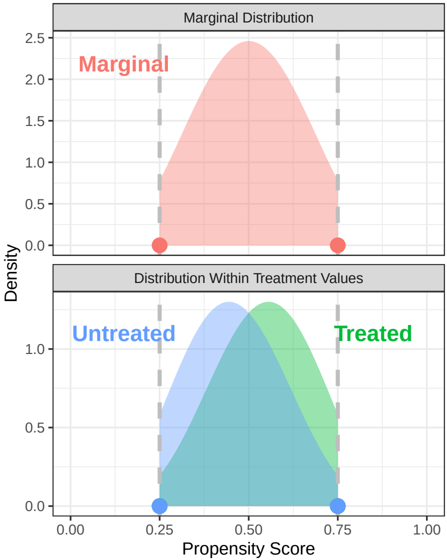


Truncate values of extreme weights

Accepting bias to reduce variance: Weight truncation



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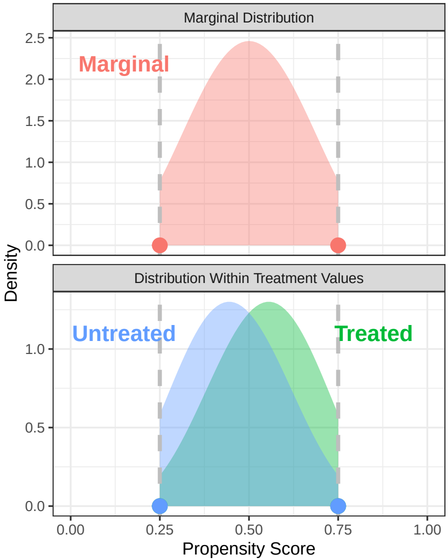

Truncate values of extreme weights

Biased: Ignores some confounding